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On The Kepler Problem with Antigravity

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Abstract

Modification of Newton's gravitational potential by an additional term containing the cosmological Lambda-term leads to the need to consider antigravity, the action of which at large distances between bodies is comparable to the gravitational force, and even prevails over it. It was found that the influence of the quadratic antigravitational force does not lead to the disintegration of the bound gravitational system, but precession of orbits occurs and the laws of motion along geodesics change. For particles with relativistic velocities, the regions of achievable motion and the shape of orbits become more complex and can be expressed in ultraelliptic integrals (as well as the motion of a test body in a two-center system).

Keywords: Cosmological term, antigravity, orbits

Introduction

The generalized Newtonian 2–particle potential $U_{GN}(r) = -\gamma M/r - c^2\Lambda r^2/6$ is a convex non-positive function whose graph lies in the IV-th quadrant on the coordinate plane (r, U_{GN}), with a maximum at $r = r_c$, equal to $U_{GN} = U_{GN}^{max}$, $r_c = (\frac{3\gamma M}{\Lambda c^2})^{1/3}$, $U_{GN}^{max} \equiv U_{GN}(r_c) = -\frac{m}{2}(3\gamma Mc)^{2/3}\Lambda^{1/3}$. The force of interaction between two gravitating objects of mass M (formally the center of attraction) and m (test body, by default $M \geq m$) is: $F=-\gamma Mm/r^2+\Lambda c^2 mr/3$, and its value is negative ($F<0$, prevailing attraction between bodies) for $r < r_c$ and positive ($F>0$, repulsion) for $r > r_c$. In the latter case, a (generally speaking, asymmetrical) gravitational dipole [M, m] is formed [1].

The point $r = r_c$ plays a special role in the dynamics of the dipole gravitational system; physically this is due to the stationarity of the processes in its vicinity due to the absence of a force action ($dU(r)/dr \sim 0$ at $r \in O(r_c; \delta_r)$) on the particles of the system located there, so that the structures formed earlier will be in a state of dynamic equilibrium in this vicinity for a fairly long time — this is a kind of analogue of the "Lagrange point" of a three-body system (it can be expected that some metastable set of particles will form at this point).

The motion of two material points obeying the modified Newton law of gravity is one-dimensional only in the simplest case (which corresponds to the variant $U = U_{GN}(r) = -\gamma M/r - c^2\Lambda r^2/6$). For motion in the central field, one should consider an additional variable associated with rotation, and introduce an effective potential $U \to U_{eff} \equiv U_{GN}(r) + p_\varphi^2/(2mr^2)$ (p_φ — the component of the momentum of particle

m moving in the generalized gravitational potential, associated with its angular momentum relative to the "center"). Depending on the value of p_φ, the motion of particle m will be either finite–infinite (three turning points, of which two are libration points and one stopping point), or simply infinite (one stopping point). It is easy to establish the critical value of the centrifugal momentum $p_\varphi = p_\varphi^{(crit)}$, corresponding to the disappearance of the bound state $U'_{GN}(r) = U_{GN}''(r) = 0$ (degeneration of the extremum/minimum of the effective potential to the inflection point): $p_\varphi^{(crit)} = \sqrt{\frac{3}{4}\gamma M m r_c^*}$, r_c^* is a root of the equation $U'_{eff} \equiv (\Lambda c^2 m) r^4 - (3\gamma M m) r + 3(p_\varphi^{(crit)})^2 = 0$. The resulting system of algebraic equations for the unknowns $(r_c^*, p_\varphi^{(crit)})$, subject to obvious physically determined assumptions (two real roots of the extremum equation, with only one root being positive), uniquely determines the moment of merging of the turning points of the librational motion into one (the inflection point of the effective potential).

Generalized Keplerian Dynamics

An explicit form of the dependence of time on coordinates:

$$t - t_0 = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{(2/m)(E - U_{eff}(r'))}} = \pm I_3^*(a_0, a_1, a_2, a_3, a_4)|_{r_0}^{r},$$

$$a_0 = \frac{\Lambda c^2}{3},\ \ a_1 = 0,\ \ a_2 = \frac{E}{3m},\ \ a_3 = \frac{GM}{2},\ \ a_4 = -\frac{p_\varphi^2}{m^2},$$

where $I_3^*(\dots)$ — elliptic integral 3rd kind (degenerate) in the Legendre form; for $p_\varphi \equiv= 0$ the solution is also obtained in a similar form, or in the form of a linear combination of elliptic integrals I_2, J_1 (of the 3rd kind) in the Weierstrass form. The conversion of the above dependence to the form $r = r(t)$ is analytically possible only in principle.

The dependence $\varphi = \varphi(\rho)$ is expressed explicitly through a (rather cumbersome) composition of incomplete elliptic integrals of the 1st and 3rd kind. Since this form of the trajectory is very important, we present a method for obtaining it. Let us introduce an intermediate variable $u = 1/r$, then we can write the following dependence for the trajectory (obtained by the Hamilton–Jacobi method):

$$\varphi_0 - \varphi = \int_{r_0}^{r} \frac{p_\varphi^2 u^{-2}}{m u^{-2}\sqrt{\frac{2}{m}(E + GMu + \frac{1}{6}\Lambda c^2 u^{-2} \frac{p_\varphi^2 u^2}{2m})}} =$$

$$= (2p_\varphi^2\sqrt{3}(\xi_4+\xi_1))\sqrt{\frac{(\xi_4-\xi_2)(u-\xi_1)}{(\xi_4-\xi_1)(u-\xi_2)}}(u-\xi_2)^2\sqrt{\frac{(\xi_2-\xi_1)(u-\xi_3)}{(\xi_3-\xi_1)(u-\xi_2)}}\sqrt{\frac{(\xi_2-\xi_1)(u-\xi_4)}{(\xi_4-\xi_1)(u-\xi_2)}}\times$$

$$\times(\xi_2)E_1\left(\sqrt{\frac{(\xi_4-\xi_2)(u-\xi_1)}{(\xi_4-\xi_1)(u-\xi_2)}}, \sqrt{\frac{(\xi_2-\xi_3)(\xi_4+\xi_1)}{(\xi_3+\xi_1)(\xi_4+\xi_2)}}\right) +$$

$$+(\xi_2+\xi_1)E_3\left(\sqrt{\frac{(\xi_4-\xi_2)(u-\xi_1)}{(\xi_4-\xi_1)(u-\xi_2)}}, \frac{\xi_4-\xi_1}{\xi_4-\xi_2}, \sqrt{\frac{(\xi_2-\xi_3)(\xi_4+\xi_1)}{(\xi_3+\xi_1)(\xi_4+\xi_2)}}\right)\times$$

$$\times((\xi_4-\xi_2)(\xi_2-\xi_1)\sqrt{-\Xi(u)},\ \ \Xi(u) \equiv -6\gamma u^3 + 3p_\varphi^2 u^4 - 6Emu^2 - \Lambda c^2 m,$$

where $\xi_i|_{i=1,\dots,4}$ are the roots of the equation $\Xi(u) = 0$.

Assuming that the value $\Lambda \ll 1$, we can obtain the influence of the oscillatory perturbation $\Lambda c^2 r^2/6$ on the Keplerian orbits that correspond to the case of gravity corresponding to pure attraction. To do this, we first move to the "action–angle" variables $\{J_r, J_\varphi, J_\theta; \chi_r, \chi_\varphi, \chi_\theta\}$:

$$2\pi J_r = -p_\sigma \oint \sqrt{(\varrho_1 - \varrho)(\varrho - \varrho_2)}\varrho^{-2} d\varrho,$$

$$\varrho_1 + \varrho_2 = 2Mm^2\kappa/p_\sigma,\ \ \varrho_1\varrho_2 = -2mE/p_\sigma^2,$$

where ϱ_1, ϱ_2 are the largest and smallest values, respectively ϱ,

$$\varrho_1 = (a - a\epsilon)^{-1},\ \ \varrho_1 = (a + a\epsilon)^{-1},$$

$$p_\sigma = m\sqrt{\kappa M(a - a\epsilon^2)} = J_\varphi + J_\theta,\ \ E = -\kappa Mm/(2a).$$

Next, we move on to the canonical elements of the Keplerian orbit w, v_1, v_2, J, u_1, u_2, expressed through the Delaunay variables $(w', v'_1, v'_2, J', u'_1, u'_2)$ and the quantities a, ϵ, m, M; they can be related to Napier's spherical coordinates, resulting in

$$r = a(1 + u_1/J - (2u_1/J)^{1/2}\cos(w + v_1) - (u_1/J)\cos(2w + 2v)),\ \ a = J^2/(\kappa m^2 M).$$

For the central disturbing force $\Lambda\mathcal{V}(r) = \Lambda c^2 r^2/6$ we have:

$$\mathcal{V}(r) = \mathfrak{V}(w_1, v_1, J, u_1) = \mathcal{V}(a) + \delta r\mathcal{V}'(a) + \tfrac{1}{2}(\delta r)^2\mathcal{V}''(a),$$

$$\delta r = (au_1/J - a(2u_1/J)^{1/2}\cos(w + v_1) - a(u_1/J)\cos(2w + 2v_1))_{a=J^2/(m^M\kappa)}.$$

The secular part $\mathfrak{V}(J, u_1)$ of the perturbing function $\mathfrak{V}(w_1, v_1, J, u_1)$ we have

$$\mathfrak{V}(J, u_1) = \mathcal{V}(a) + (au_1/J)(\mathcal{V}'(a) + (a/2)\mathcal{V}''(a)),$$

differential equations for secular perturbations of variables $v_1,\ u_1$:

$$dv_1/dt = \Lambda\partial\mathfrak{V}(J, u_1)/\partial u_1 = \Lambda(a/J)(\mathcal{V}'(a) + (a/2)\mathcal{V}''(a)),\ \ du_1/dt = 0.$$

Under the perturbation of the central force $\sim \Lambda r^2$ the Keplerian ellipse retains its size and shape on average over the period. It experiences as a whole only slowly changing motion: $v_1 = v_1^{(0)} + \omega t$, $\omega = \omega_0 \equiv \Lambda(a/J)(\mathcal{V}'(a) + (a/2)\mathcal{V}''(a))$. According to the value of the Delaunay variable $\Pi = -v_1$, which means that the line of apsides of the Kepler ellipse (the direction to the perihelion) undergoes slow rotations with an angular velocity of ω, while $u_2 = const_1$, $v_2 = const_2$. Since the latter quantities are related to the orbital inclination, this means that the central perturbing force leaves the motion flat (in the Laplace plane). The energy of the perturbed motion $E(J, u) = -\kappa^2 M^2 m^2/(2J^2) + \omega u\ (\omega \equiv \omega_0)$.

Let us consider the motion of a test body P (mass m) in the gravitational field of two fixed centers K, K' of mass $M(\gg m)$: $|KP| = r, |K'P| = r'$. In this case, it is advisable to place the origin of coordinates between tt. K, K' ($|KO| = |OK'| = c$), and introduce elliptical coordinates $\lambda = (r + r')/2$, $\mu = (r - r')/2$. The integral of living forces then takes the form

$$T = \frac{1}{2}(\lambda^2 - \mu^2)(\frac{(\dot{\lambda})^2}{\lambda^2 - c^2} + \frac{(\dot{\mu})^2}{c^2 - \mu^2}),$$

and the potential (force function):

$$U = -\gamma M(\frac{1}{\lambda+\mu} - \frac{1}{\lambda-\mu}) - \frac{\Lambda c^2}{6}(\lambda^2 + \mu^2) = -2\gamma M\frac{\lambda-\mu}{\lambda^2-\mu^2} - \frac{c^2\Lambda(\lambda^4-\mu^4)}{6(\lambda^2-\mu^2)}.$$

The conditions of Liouville's theorem are satisfied, since the integral of living forces corresponds in form to the conditions of the theorem: $2T = (\sum_{1,2}\phi_i)(\sum_j A_j(q_j)\dot{q}_j^2)$, and the potential, which must have the form $U(q_1, q_2) = (\psi_1(q_1) + \psi_2(q_2))/(\phi_1(q_1) + \phi_2(q_2))$, is represented as:

$$U(\lambda,\mu) = \frac{-2M\gamma(\lambda-\mu) - (c^2\Lambda/6)(\lambda^4-\mu^4)}{\lambda^2-\mu^2}$$
$$= \frac{-2M\gamma\lambda - (c^2\Lambda/6)\lambda^4 + 2M\gamma\mu + (c^2\Lambda/6)\mu^4}{\lambda^2-\mu^2}.$$

Therefore, the Lagrange differential equations in this case are integrated in quadratures, having the form

$$\int\sqrt{A_1/F_1}dq_1 + \beta_1 = \int\sqrt{A_2/F_2}dq_2 + \beta_2,$$
$$\int\sqrt{A_1/F_1}\phi_1 dq_1 + \int\sqrt{A_2/F_2}\phi_2 dq_2 = \sqrt{2}t + C,$$

where $F_j = \psi_j(q_j) + h\phi_j(q_j) + \alpha_j,\ \beta_j, h, \alpha_j$ — constants. In our case we get

$$\frac{d\lambda}{(\lambda^2-c^2)(-2M\gamma\lambda - c^2\Lambda\lambda^4/6 + h\lambda^2 + \alpha)} - \frac{d\mu}{(c^2-\mu^2)(2M\gamma\mu + c^2\Lambda\mu^4/6 + h^2\mu + \alpha)} = 0,$$

$$\frac{\lambda^2 d\lambda}{(\lambda^2-c^2)(-2M\gamma\lambda - c^2\Lambda\lambda^4/6 + h\lambda^2 + \alpha)} - \frac{\mu^2 d\mu}{(c^2-\mu^2)(2M\gamma\mu + c^2\Lambda\mu^4/6 + h^2\mu + \alpha)} = \sqrt{2}t,$$

in this case, how does it happen that λ, μ can be represented as four-periodic functions of two arguments.

Conclusions

Integration up to quadratures of these equations can, for example, demonstrate to us the behavioral features of the Magellanic Clouds — satellite galaxies of our Milky Way Galaxy [2], also located in the zone of influence of the Andromeda Nebula, or our local cluster as an object of comparatively small mass in the field of the gravitational dipole of the Shapley and Repeller superclusters.

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