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Approximate Analytical Expressions of Non-Linear Boundary Value problem for a Boundary Layer Flow using the Homotopy Analysis Method

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Abstract

In this paper, we discuss the analytical expressions of velocity and heat transfer for a boundary layer flow with thermal radiation past a moving vertical porous plate. The governing non-linear differential equations are solved analytically using Homotopy analysis method. The results are presented as velocity, temperature, local wall shear stress and wall heat transfer rate profiles various values of parameter involving in the problem. Our analytical results are compared with the previous work and a good agreed is observed.

Keywords: Thermal Radiation; Porous Plate; Non-Linear Boundary Value Problem; Local Wall Shear Stress; Wall Heat Transfer Rate; Homotopy Analysis Method.

Introduction

The study of the flow and heat transfer in fluid past a porous surface has many interesting applications in the field of engineering, especially in physical and chemical industries [1-6]. Makinde et al. [1] investigated the Adomian decomposition approach to a boundary layer flow with thermal radiation past a moving vertical porous plate. Adomian [3] examined a review of the decomposition method in applied mathematics. Adomian et al. [4] investigated the solution of nonlinear ordinary and partial differential equations of physics. Chung et al. [7] examined the stability of steady flow in a channel with linear temperature dependent viscosity. Hayat et al. [8] investigated the hydromagnetic oscillatory flow of a fluid bounded by a porous plate when the entire system rotates about axis normal to the plate.

Hassanien [9] examined the heat transfer in a power law fluid over a non-isothermal stretching sheet. Howell et al. [10] investigated momentum and heat transfer on a continuously moving surface in a power law fluid. Ibrahim et al. [11] examined the method of similarity reduction for problems of radiative and magnetic field effect on free convection and mass transfer flow past a semi-infinite flat plate. Makinde [12] examined free convection flow with thermal radiation and mass transfer past a moving vertical porous plate. Sivasankaran et al. [13] investigated the natural convection heat and mass transfer fluid past an inclined semi-infinite porous surface. Terrill [14] examined slow laminar flow in a converging or diverging channel with suction at one wall and blowing at the other wall. Uzan [15] investigated the heat transfer to a power law fluid in arbitrary cross sectional ducts. Yu-shu et al. [16] examined a numerical for simulating non-Newtonian fluid flow and displacement in porous media. Badr et al [5] investigated

the numerical simulation of steady and unsteady mixed convection from tubes of elliptic cross-section. The study revealed that the effect of fluctuations on the time-average Nusselt number becomes more pronounced with increasing Reynolds number. It also revealed that the rate of heat transfer increases with the increase of the amplitude of fluctuations but decreases with the increase of frequency. Yurusoy et al. [17] examined the exact solution of boundary layer equations of a non-Newtonian fluid over a stretching sheet by the method of lie group analysis. This paper demonstrates the solution of the velocity and heat transfer in a boundary layer flow with thermal radiation past a moving vertical porous plate using Homotopy analysis method and discussed by graphically.

Mathematical Formulation of the Problem

Consider an unsteady flow of an incompressible fluid with thermal radiation past a moving vertical plate. Let the x-axis be taken along the plate in the vertically upwards direction and the y-axis be taken normal to it. Let u and v be the velocity components along the x and y respectively. The physical variables are functions of y and t only. Hence, the governing equations are as follows:

$$\frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \tag{2}$$

$$\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\alpha}{k} \frac{\partial q_r}{\partial y} \tag{3}$$

The radiative heat flux term is simplified by making use of the Roseland approximation as in eqn (1).

$$q_r = -\frac{4\sigma}{3\delta} \frac{\partial T^4}{\partial y} \tag{4}$$

Where, α denote thermal diffusivity, k is thermal conductivity, σ denote Stefan–Boltzmann constant, K is absorption Coefficient, ν is Kinematics viscosity, u, v denote velocity component, q_r is Radiative heat flux, β is Volumetric expansion coefficient(Temperature), G denote Gravitational acceleration, t is Time, U_0 denote wall velocity, T_w is Wall Temperature, T_∞ denote ambient temperature, c is suction parameter, R is radiation parameter, G_r denote local Grash of number, P_r is Prandtl number.

The appropriate boundary conditions are

$$\begin{aligned} u = U_0, T = T_w \text{ at } y' = 0 \\ u \rightarrow 0, T \rightarrow T_\infty \text{ at } y' \rightarrow \infty \end{aligned} \tag{5}$$

We can define the similarity variables are as follows:

$$y = \frac{y'}{\sqrt{\nu t}} \tag{6}$$

Here the length scale is defined as

$$\delta = 2\sqrt{\nu t}, u = U_0 f(y), \theta(y) = \frac{T - T_\infty}{T_w - T_\infty} \tag{7}$$

Taking the Taylor series expansion of T^4 and neglecting terms with higher powers, we have

$$T^4 \approx 4T_\infty^3 - 3T_\infty^4 \tag{8}$$

Using the eqns. (7) and (8) into the eqns. (2) to (4) we obtained the non-linear ordinary differential equations are as follows:

$$f'' = -2(c + y)f' - G_r \theta \tag{9}$$

$$\theta'' = -\frac{2P_r}{(1 + R)}(c + y)\theta' \tag{10}$$

The corresponding boundary conditions are as follows:

$$f(0) = \theta(0) = 1 \text{ and } f(\infty) = \theta(\infty) = 0 \tag{11}$$

Here

$$R = \frac{16\sigma\alpha T_\infty^3}{3kK} \quad G_r = \frac{g\beta(T_w - T_\infty)\delta^2}{\nu U_0} \quad P_r = \frac{\nu}{\alpha} \tag{12}$$

The local wall shear stress (Skin friction) can be defined as

$$\tau_w = \left[\mu \frac{\partial u}{\partial y} \right]_{y=0} = \frac{\tau_w \delta}{\mu U_0} = f'(0) \tag{13}$$

The Local surface heat flux

$$q_w = \left[-k \frac{\partial T}{\partial y} \right]_{y=0} - \left[\frac{4\sigma}{3K} \frac{\partial T^4}{\partial y} \right]_{y=0} \tag{14}$$

$$Nu = \frac{q_r \delta}{k(T_w - T_\infty)(1 + R)} = -\theta'(0) \tag{15}$$

Solution of the Non-Linear Boundary problem using the Homotopy Analysis Method

This section deals with a basic strong analytic tool for non-linear problems, namely the Homotopy analysis method (HAM) which was generated by Liao [18], is employed to solve the nonlinear differential eqns. (9)–(11). The Homotopy analysis method is based on a basic concept in topology. Unlike perturbation techniques like [19], the Homotopy analysis method is independent of the small/large parameters. Unlike all other reported perturbation and non-perturbation techniques such as the artificial small parameter method [20], the δ -expansion method [21] and Adomian’s decomposition method [22], the Homotopy analysis method provides us a simple way to adjust and control the convergence region and rate of approximation series. The Homotopy analysis method has been successfully applied to many nonlinear problems such as heat transfer [23], viscous flows [24], nonlinear oscillations [25], Thomas-Fermi’s atom model [26], nonlinear water waves [27], etc. Such varied successful applications of the Homotopy analysis method confirm its validity for nonlinear problems in science and engineering. The Homotopy analysis method is a good technique when compared to other perturbation methods. The existence of the auxiliary parameter h in the Homotopy analysis method provides us with a simple way to adjust and control the convergence region of the solution series.

Basic concepts of the Homotopy analysis method [18-29]

Consider the following differential equation:

$$N[u(t)] = 0 \tag{16}$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao constructed the so-called zero-order deformation equation as:

$$(1 - p)L[\varphi(t; p) - u_0(t)] = phH(t)N[\varphi(t; p)] \tag{17}$$

Where $p \in [0, 1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t; p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds:

$$\varphi(t; 0) = u_0(t) \text{ and } \varphi(t; 1) = u(t) \tag{18}$$

respectively. Thus, as P increases from 0 to 1, the solution $\varphi(t; p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$.

Expanding $\varphi(t; p)$ in Taylor series with respect to p , we have:

$$\varphi(t; p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m \tag{19}$$

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t; p)}{\partial p^m} \tag{20}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h , and the auxiliary function are so properly chosen, the series eqn.(18) converges at $p=1$ then we have:

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) \tag{21}$$

Differentiating the eqn. (16) for m times with respect to the embedding parameter p , and then setting $p=0$ and finally dividing them by $m!$, we will have so-called m th order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\mathfrak{R}_m \left(\vec{u}_{m-1} \right) \tag{22}$$

Where

$$\mathfrak{R}_m \left(\vec{u}_{m-1} \right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t; p)]}{\partial p^{m-1}} \tag{23}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{24}$$

Applying L^{-1} on both side of eqn.(22), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1} \left[H(t)\mathfrak{R}_m \left(\vec{u}_{m-1} \right) \right] \tag{25}$$

In this way, it is easily to obtain u_m for $m \geq 1$, at M th order, we have

$$u(t) = \sum_{m=0}^M u_m(t) \tag{26}$$

when $M \rightarrow +\infty$, we get an accurate approximation of the original eqn.(16). For the convergence of the above method we refer the reader to Liao [19]. If an eqn. (16) admits unique solution, then this method will produce the unique solution.

Approximate analytical expressions of the non-linear differential eqns. (9) and (10) using Homotopy analysis method

In this section, we find the analytical expressions for the eqns. (9) and (10) with the help of the eqn.(10). We construct Homotopy for the eqns.(9) and (10) are as follows:

$$(1 - p) \left(\frac{d^2 f}{dy^2} + 2c \frac{df}{dy} \right) - h p \left(\frac{d^2 f}{dy^2} + 2c \frac{df}{dy} + 2y \frac{df}{dy} + G_r \theta \right) = 0 \tag{27}$$

$$(1 - p) \left(\frac{d^2 \theta}{dy^2} + ac \frac{d\theta}{dy} \right) - h p \left(\frac{d^2 \theta}{dy^2} + ac \frac{d\theta}{dy} + ay \frac{d\theta}{dy} \right) = 0 \tag{28}$$

The approximate solution of the eqns.(27) and (28) are as follows:

$$u = f_0 + fp_1 + p^2 f_2 + p^3 f_3 + \dots \tag{29}$$

$$\theta = \theta_0 + p\theta_1 + p^2 \theta_2 + p^3 \theta_3 + \dots \tag{30}$$

The initial approximations are as follows:

$$f_0(0) = \theta_0(0) = 1 \tag{31}$$

$$f_i(0) = \theta_i(0) = 0, \quad i = 1, 2, 3, \dots \tag{32}$$

$$f_0(\infty) = \theta_0(\infty) = 0 \tag{33}$$

$$f_i(\infty) = \theta_i(\infty) = 0, \quad i = 1, 2, 3, \dots \tag{34}$$

Substituting the eqns. (29) and (30) into the eqns. (27) and (28) respectively we get

$$(1 - p) \left(\frac{d^2}{dy^2} (f_0 + pf_1 + \dots) + 2c \frac{d}{dy} (f_0 + pf_1 + \dots) \right) - hp \left(\frac{d^2}{dy^2} (f_0 + pf_1 + \dots) + 2c \frac{d}{dy} (f_0 + pf_1 + \dots) + 2y \frac{d}{dy} (f_0 + pf_1 + \dots) + G_r \theta \right) = 0 \tag{35}$$

$$(1 - p) \left(\frac{d^2}{dy^2} (\theta_0 + p\theta_1 + \dots) + ac \frac{d}{dy} (\theta_0 + p\theta_1 + \dots) \right) - hp \left(\frac{d^2}{dy^2} (\theta_0 + p\theta_1 + \dots) + ac \frac{d}{dy} (\theta_0 + p\theta_1 + \dots) + ay \frac{d}{dy} (\theta_0 + p\theta_1 + \dots) \right) = 0 \tag{36}$$

Comparing the coefficients of p^0, p^1 in the eqns.(35) and (36) we get

$$p^0 : \frac{d^2 \theta_0}{dy^2} + ac \frac{d\theta_0}{dy} = 0 \tag{37}$$

$$p^1 : \frac{d^2 \theta_1}{dy^2} + ac \frac{d\theta_1}{dy} + ay \frac{d\theta_0}{dy} = 0 \tag{38}$$

$$p^0 : \frac{d^2 f_0}{dy^2} + 2c \frac{df_0}{dy} = 0 \tag{39}$$

$$p^1 : \frac{d^2 f_1}{dy^2} + 2c \frac{df_1}{dy} + 2y \frac{df_0}{dy} + G_r \theta_0 = 0 \tag{40}$$

Solving the eqns. (37)-(40) with the help of the eqns. (31)-(34) we get the following results:

$$\theta_0 = e^{-acy} \tag{41}$$

$$\theta_1 = -ae^{-acy} \left(\frac{y^2}{2} + \frac{y}{ac} \right) \tag{42}$$

$$f_0 = e^{-2cy} \tag{43}$$

$$f_1 = \left(\frac{G_r}{a^2 c^2 - 2ac^2} \right) e^{-2cy} - 2e^{-2cy} \left(\frac{y^2}{2} + \frac{y}{2c} \right) - \left(\frac{G_r}{a^2 c^2 - 2ac^2} \right) e^{-acy} \tag{44}$$

Where

$$a = \frac{2p_r}{(1 + R)} \tag{45}$$

According to the HAM, we can conclude that

$$f = \lim_{p \rightarrow 1} f(y) = f_0 + f_1 \tag{46}$$

$$\theta = \lim_{p \rightarrow 1} \theta(y) = \theta_0 + \theta_1 \tag{47}$$

Substituting the eqns. (43) and (44) into an eqn. (46) and using the eqns. (41) and (42) into an eqn. (47) we get the following:

$$\theta = e^{-acy} + ha e^{-acy} \left(\frac{y^2}{2} + \frac{y}{ac} \right) \tag{48}$$

$$f = e^{-2cy} - h \left[\frac{G_r}{a^2c^2 - 2ac^2} e^{-2cy} - 2e^{-2cy} \left(\frac{y^2}{2} + \frac{y}{2c} \right) - \frac{G_r e^{-acy}}{a^2c^2 - 2ac^2} \right] \tag{49}$$

The analytical expression of the dimensionless skin friction using the eqn. (13) is given by Skin friction

$$= f'(0) = - .5 \frac{4P_r c^2 - 2hP_r + hG_r R + hG_r}{cP_r} \tag{50}$$

The analytical expression of the Wall heat transfer rate using the eqn. (15) is given by

$$Nu = -\theta'(0) = - \frac{2P_r c^2 - h - hR}{(1 + R)c} \tag{51}$$

Results and Discussion

Figures 1 and 2 represents dimensionless temperature $\theta(y)$ versus dimensionless distance y . From figure 1, it is noted that the temperature increases when the radiation parameter R increases, and in some fixed values of the other dimensionless parameters. From figure 3, it is inferred that when the suction parameter c increases the corresponding temperature decreases in some fixed values of the other parameters.

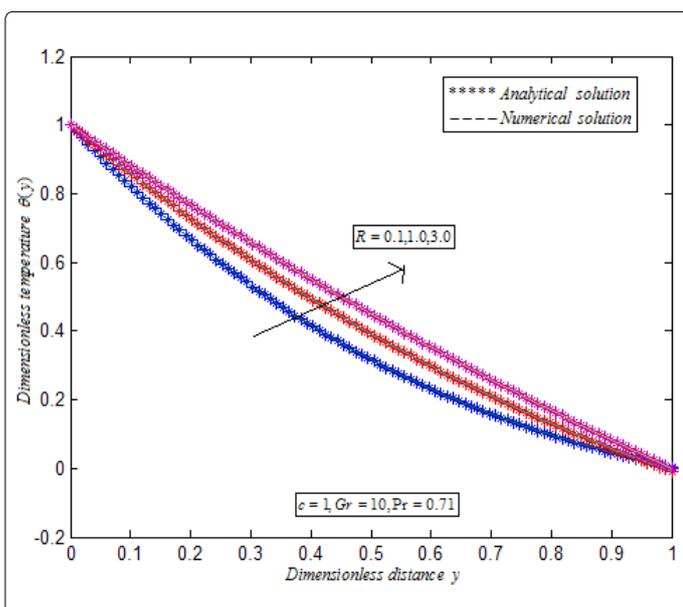


Figure 1. Dimensionless temperature $\theta(y)$ versus the dimensionless distance y . The curves are plotted using the eqn.(48) for various values of the radiation parameter R , and in some fixed values of the other dimensionless parameters c, Gr, Pr .

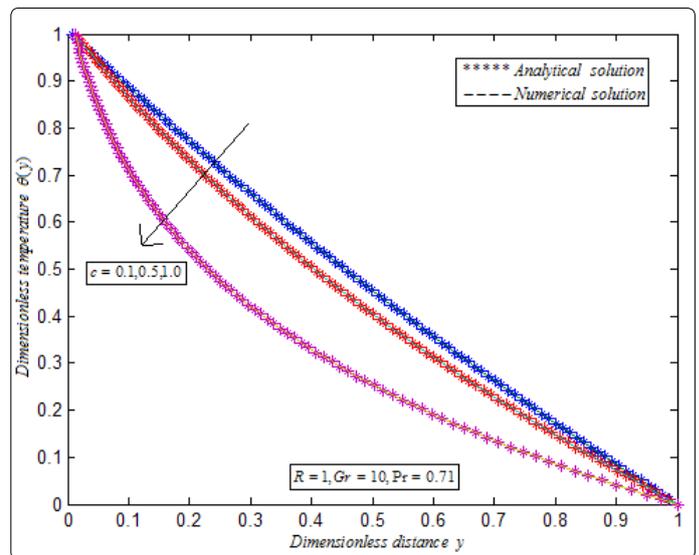


Figure 2. Dimensionless temperature $\theta(y)$ versus the dimensionless distance y . The curves are plotted using the eqn. (48) for various values of the suction parameter c , and in some fixed values of the other dimensionless parameters R, Gr, Pr .

Figures 3 and 4 represent dimensionless velocity $f(y)$ versus dimensionless distance y . From figure 3, it is noted that the velocity increases when the radiation parameter R increases, and in some fixed values of the other dimensionless parameters. From figure 4, it is inferred that when the suction parameter c increases the corresponding velocity decreases in some fixed values of the other parameters.

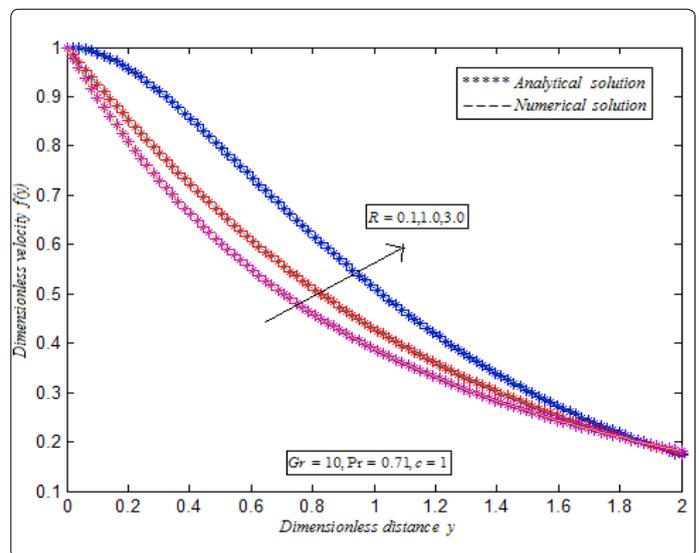


Figure 3. Dimensionless velocity $f(y)$ versus the dimensionless distance y . The curves are plotted using the eqn.(47) for various values of the radiation parameter R , and in some fixed values of the other dimensionless parameters c, Gr, Pr .

Figure 5 represents wall heat transfer rate Nu versus radiation parameter R . From figure 5, it is observed that the wall heat transfer rate increases when the suction parameter increases, and in some fixed values of the other dimensionless parameters. Figure 6 represents wall shear stress versus radiation parameter R . From figure 6, it is inferred that the wall shear stress decreases when the suction parameter increases, and in some fixed values of the other dimensionless parameters.

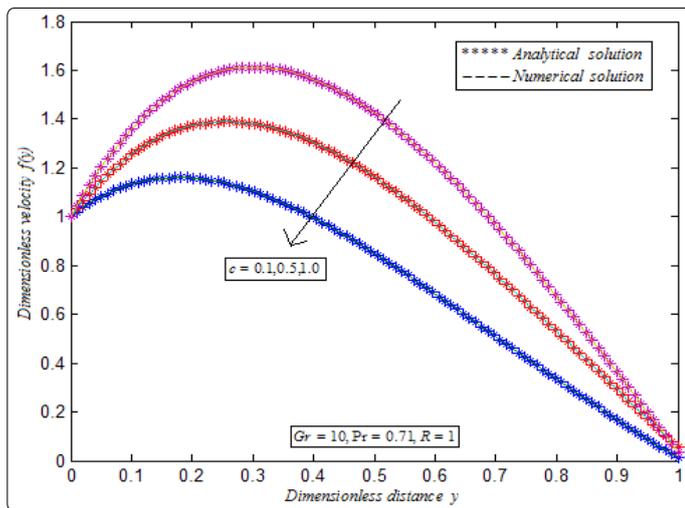


Figure 4. Dimensionless velocity $f(y)$ versus the dimensionless distance y . The curves are plotted using the eqn.(47) for various values of the suction parameter c , and in some fixed values of the other dimensionless parameters R , Gr , Pr .

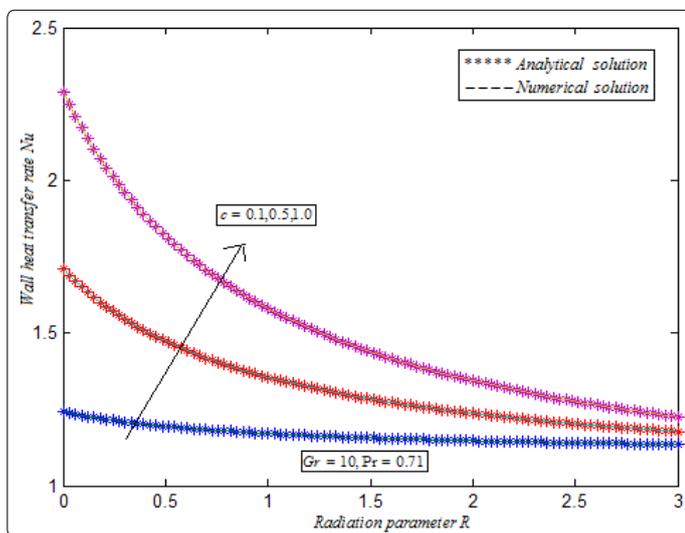


Figure 5. Dimensionless wall heat transfer rate Nu versus the Radiation parameter R . The curves are plotted using the eqn. (50) for various values of the suction parameter c , and in some fixed values of the other dimensionless parameters Gr , Pr .

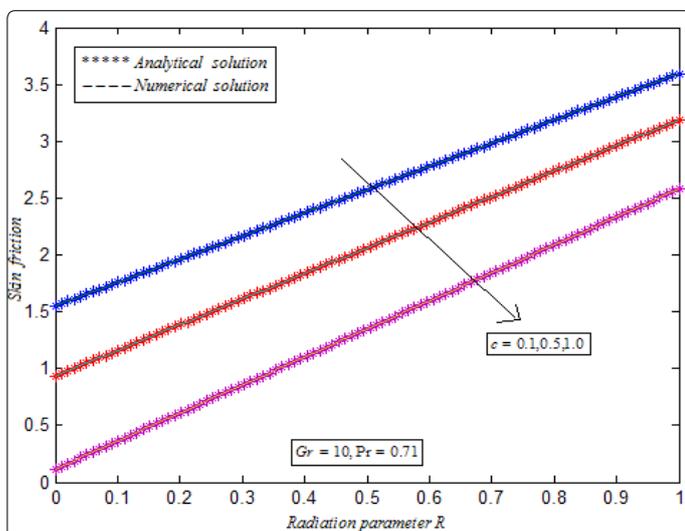


Figure 6. Dimensionless wall shear stress versus the Radiation parameter R . The curves are plotted using the eqn. (49) for various values of the suction parameter c , and in some fixed values of the other dimensionless parameters Gr , Pr .

Conclusion

In this paper the Homotopy analysis method is employed to mathematical study of a boundary layer flow with thermal radiation past a moving vertical porous plate. The approximate analytical expressions of the velocity and temperature profiles are derived mathematically and graphically using the Homotopy analysis method. The approximate analytical expressions of the wall shear stress and wall heat transfer rate are also derived using the analytical expressions for the velocity and temperature profiles. We also discussed the graphical representations of the wall shear stress and the wall heat transfer rate. The Homotopy analysis method can be easily extended to solve the other non-linear boundary value problems in physical and chemical sciences.

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Appendix: Nomenclature

Symbol	Meanings
α	Thermal diffusivity
K	Thermal conductivity
σ	Stefan-Boltzmann constant
K	Absorption Coefficient
ν	Kinematics viscosity
u, v	Velocity component
q_r	Radiative heat flux
β	Volumetric expansion coefficient
G	Gravitational acceleration
t	Time
U_0	Wall velocity
T_w	Wall Temperature
T_∞	Ambient temperature
C	Suction parameter
R	Radiation parameter
G_r	Local Grashof number
P_r	Prandtl number